On Certain Generalized $\mathcal{B}K$ −Recurrent Affinely Connected and Landsberg Spaces

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Abstract
In this paper we introduced a generalized $\mathcal{B}K$-recurrent space for which Cartan's fourth curvature tensor K^i_{jkh} satisfies the generalized recurrence property, i.e. characterized by the condition

$$\mathcal{B}_m K^i_{jkh} = \lambda_m K^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) , \qquad K^i_{jkh} \neq 0 ,$$

Which is also a finely connected space and Landsberg space, separately, where $\mathcal{B}_m$ is covariant differential operator with respect to x^m in the sense of Berwald, such spaces called a generalized $\mathcal{B}K$ -recurrent affinely connected space and generalized $\mathcal{B}K$ -recurrent Landsberg space, respectively.
The purpose of this paper to develop some properties of generalized $\mathcal{B}K$-recurrent affinely connected space and generalized $\mathcal{B}K$-recurrent Landsberg space by obtaining the condition for some tensors to possess the properties of these spaces and to obtain various identities in such space.

Keywords: *Generalized $\mathcal{B}K$-recurrent affinely connected space, Generalized $\mathcal{B}K$-recurrent Landsberg space.*

1. Introduction

H. D. Pande and B. Singh [11] discussed the recurrence condition in an affinely connected space. A. A. A. Muhib [10] obtained some results when R^h- generalized trirecurrent and R^h- special generalized trirecurrent space are affinely connected spaces. Landsberg space of dimension 2 was first considered by G. Landsberg [7] from a standpoint of variation. Also such spaces of many dimension, É. Cartan [3] introduced it as one of particular cases and further L. Berwald ([1], [2]) showed that the space was characterized by $P^i_{jkh} = 0$, where P^i_{jkh} is the (hv) curvature tensor. H. Yasuda [16] gave other characterization of Landsberg space and contributed a little for the theory of Landsberg space.

Let us consider an n-dimensional Finsler space F_n equipped with the metric function F(x,y) satisfies the request conditions [14].

The relation between the metric function F and the corresponding metric tensor given by

(1.1) $\qquad g_{ij}(x,y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2(x,y)$.

The metric tensor g_{ij} satisfies the following relations

(1.2) $\qquad$ a) $g_{ij}(x,y)\, y^i = y_j$ $\qquad$ and $\qquad$ b) $\delta^i_h g_{ik} = g_{hk}$.

The tensor $g_{ij}(x,y)$ is symmetric and positively homogeneous of degree zero in y^i.

The two sets of quantities g_{ij} and its associative g^{ij}, which are components of a metric tensor are connected by

(1.3) $\qquad g_{ij}g^{ik} = \delta^k_j = \begin{cases} 1 & if\ i = k, \\ 0 & if\ i \neq k \end{cases}$

By differentiating (1.1) partially with respect to y^k, we construct a new tensor C_{ijk} is defined by

$$C_{ijk} = \frac{1}{2}\dot{\partial}_i g_{jk}.$$

This new tensor C_{ijk} is positively homogeneous of degree -1 in y^i and symmetric in all its indices and called *(h)hv-torsion tensor* [8]. A according to Euler's theorem on homogeneous functions, this tensor satisfy the following:

(1.4) $a)\ C_{ijk}y^i = C_{kij}y^i = C_{jki}y^i = 0$,

$b)\ C_{jsk} = C^i_{jk}\, g_{is}$

and

$c)\ C_{jsk}g^{ji} = C^i_{sk}.$

Berwald's covariant derivative of the vector y^i vanish identically, i.e.

(1.5) $\mathcal{B}_k y^i = 0.$

In general, Berwald's covariant derivative of the metric tensor g_{ij} doesn't vanish and given by

(1.6) $\mathcal{B}_k g_{ij} = -2C_{ijk|h}y^h = -2y^h\mathcal{B}_h C_{ijk}.$

The curvature tensor R^i_{jkh} is called *Cartan's third curvature tensor*, it is positively homogeneous of degree zero in y^i, which defined by

$$R^i_{jkh} := \partial_h\Gamma^{*i}_{kj} + (\dot{\partial}_l\Gamma^{*i}_{jk})G^l_{kh} + G^i_{jm}(\dot{\partial}_k G^m_h - G^m_{kl}G^l_h) + \Gamma^{*i}_{mk}\Gamma^{*m}_{jh} - k/h\,^*.$$

The associate curvature tensor R_{ijkh} of the curvature tensor R^i_{jkh} is given by

(1.7) $R_{ijkh} \coloneqq g_{rj}R^r_{ikh}.$

The associateive curvature tensor R_{ijkh} satisfies the following relation

(1.8) $R_{ijkh} = K_{ijkh} + C_{ijs}H^s_{kh}.$

The associate curvature tensor K_{ijkh} of the curvature tensor K^i_{jkh} is given by

(1.9) $K^r_{ikh}g_{rj} = K_{jikh}.$

Cartan's third curvature tensor R^i_{jkh}, Cartan's fourth curvature tensor K^i_{jkh} and the h(v)-torsion tensor H^i_{kh} are related by

(1.10) $R^i_{jkh}y^j = H^i_{kh} = K^i_{jkh}y^j.$

The h(v)-torsion tensor H^i_{kh} and the deviation tensor H^i_h are positively homogeneous of degree one and two in y^i, respectively. In view of Euler's theorem on homogeneous functions and since the contraction of indies doesn't effect on the degree of the homogeneous, we have the following relation

* $-k/h$ means the subtraction from the former term by interchanging the indices k and h .

(1.11) $H^i_{kh}y^k = H^i_h = -H^i_{hk}y^k.$

A Finsler space F_n for which the curvature tensor R^i_{jkh} satisfy the following [15] :

(1.12) $\mathcal{B}_m R^i_{jkh} = \lambda_m R^i_{jkh}$, $R^i_{jkh} \neq 0$

is called R^h *−recurrent space*, where λ_m and a_{lm} are non-zero covariant vector field and covariant tensor field, respectively.

Transvecting the condition (1.12) by y^j , using (1.5) and (1.10), we get

(1.13) $\mathcal{B}_m H^i_{kh} = \lambda_m H^i_{kh}.$

Let the current coordinates in the tangent space at the point x_0 be x^i, then the indicatrix I_{n-1} is a hypersurface defined by

(1.14) $F(x_0\,,\, x^i) = 1$

or in the parametric form it is defined by

(1.15) $x^i = x^i(u^a)$, $a = 1,2,\ldots,n-1.$

Now, the projection of any tensor T_j^i on the indicatrix is given by

(1.16) $p.T_j^i := T_b^a\ h_a^i\, h_j^b,$

where

(1.17) $h_c^i := \delta_c^i - l^i l_c\,.$

2. A Generalized $\mathcal{B}$K –Recurrent Space

Let us consider a Finsler space F_n for which Cartan's fourth curvature tensor K_{jkh}^i satisfies the condition [12]

(2.1) $\mathcal{B}_m K_{jkh}^i = \lambda_m K_{jkh}^i + \mu_m(\delta_h^i g_{jk} - \delta_k^i g_{jh})$, $K_{jkh}^i \neq 0$,

is called *a generalized $\mathcal{B}K$-recurrent space* and briefly denoted by $G\mathcal{B}K - RF_n$ where λ_m and μ_m are covariant vectors field.

We have the condition [13]

(2.2) $\mathcal{B}_m R_{jkh}^i = \lambda_m R_{jkh}^i + \mu_m(\delta_h^i g_{jk} - \delta_k^i g_{jh} + C_{jh}^i y_k - C_{jk}^i y_h) + (\mathcal{B}_m C_{jr}^i) H_{kh}^r.$

3. A Generalized $\mathcal{B}$K – Recurrent Affinely Connected Space

We shall introduce definition for a generalized $\mathcal{B}$K-recurrent space to be also affinely connected space.

Definition 3.1. A Finsler space whose Berwald's connection parameter G_{kh}^i is independent of y^i is called *an affinely connected space* (*Berwald*'s *space*).

Thus, an affinely connected space is characterized by one of the equivalent conditions

(3.1) $a)\ G_{jkh}^i = 0$ and $b)\ C_{ijk|h} = 0$.

Remark 3.1. The connection parameter Γ_{kh}^{*i} of É. Cartan and G_{kh}^i of L. Berwald coincide in affinely connected space and they are independent of the direction arguments [14], i.e.

(3.2) $a)\ G_{jkh}^i = \dot{\partial}_j G_{kh}^i = 0$ and $b)\ \dot{\partial}_j \Gamma_{kh}^{*i} = 0.$

Remark 3.2. In particular, the metric tensor g_{ij} and its associative g^{ij} are covariant constants in the sense of Berwald for affinely connected space, i. e.

(3.3) $a)\ \mathcal{B}_k g_{ij} = 0$ and $b)\ \mathcal{B}_k g^{ij} = 0.$

Definition 3.2. The generalized $\mathcal{B}$K-recurrent space which is affinely connected space [satisfies any one of the conditions (3.1a), (3.1b), (3.2a) and (3.2b)], will be called *a generalized $\mathcal{B}K$-recurrent affinely connected space* and will be denote it briefly by $G\mathcal{B}K - R -$ *affinely connected space.*

Remark 3.2. It will be sufficient to call the tensor which satisfies the condition of G$\mathcal{B}$K – R –affinely connected space as *generalized $\mathcal{B}$-recurrent tensor* (briefly $G\mathcal{B} - R$).

Let us consider a G$\mathcal{B}$K – R – affinely connected space.

Transvecting the condition (2.2) by g_{ip} , using (1.7) and (3.3a), we get

$$\mathcal{B}_m R_{jpkh} = \lambda_m R_{jpkh} + g_{ip}\mu_m(\delta_h^i g_{jk} - \delta_k^i g_{jh} + C_{jph} y_k - C_{jpk} y_h) + (\mathcal{B}_m C_{jpr}) H_{kh}^r\,.$$

This shows that

(3.4) $\mathcal{B}_m R_{jpkh} = \lambda_m R_{jpkh}$

if and only if

(3.5) $g_{ip}\mu_m(\delta_h^i g_{jk} - \delta_k^i g_{jh} + C_{jph} y_k - C_{jpk} y_h) + (\mathcal{B}_m C_{jpr}) H_{kh}^r = 0.$

Thus, we conclude

Theorem 3.1. *In $G\mathcal{B}K-R-$ affinely connected space, the associative curvature tensor R_{jpkh} behaves as recurrent if and only if (3.5) holds good.*

Transvecting the condition (2.2) by y^j , using (1.10), (1.5), (1.2a) and in view of (1.4a), we get

(3.6) $\mathcal{B}_m H^i_{kh} = \lambda_m H^i_{kh} + \mu_m(\delta^i_h y_k - \delta^i_k y_h)$.

Thus, we conclude

Theorem 3.2. *In $G\mathcal{B}K-RF_n$, Berwald's covariant derivative of first order for the h(v) - torsion tensor H^i_{kh} is given by the condition (3.6).*

By using (3.3b), the equation [13]

$$\mathcal{B}_m K = \lambda_m K + n(n-1)\mu_m + (\mathcal{B}_m g^{jk})k_{jk}$$

can be written as

$$\mathcal{B}_m K = \lambda_m K + n(n-1)\mu_m \ .$$

Thus, we conclude

Theorem 2.3.4. *In $G\mathcal{B}K-RF_n$, the curvature scalar K (of Cartan's fourth curvature tensor K^i_{jkh}),* is non vanish.

4. A Generalized $\mathcal{B}$K − Recurrent Landsberg Space

Cartan's connection parameter Γ^{*i}_{kh} coincided with Berwald's connection parameter G^i_{kh} for a Landsberg space which characterized by the condition

(4.1) $y_r G^r_{ijk} = -2C_{ijk|h}y^h = -2P_{ijk} = 0$.

Various authors denote the tensor $C_{jkh|l}y^l$ by P_{jkh} (H.Izumi [4], [5], [6]), and (M. Matsumoto [9]).

Remark 4.1. In view of the conditions (3.1a), (3.1b), (3.2a) and (4.1), an affinely connected space is necessarily a Landsberg space. However, a Landsberg space need not be an affinely connected space.

Definition 4.1. The generalized $\mathcal{B}K$-recurrent space which is Landsberg [satisfies the conditions (4.1)], will be called *a generalized $\mathcal{B}K$-recurrent Landsberg space* and will be denote it briefly by *$G\mathcal{B}K-R-$ Landsberg space.*

Remark 4.1. It will be sufficient to call the tensor which satisfies the condition of G$\mathcal{B}$K − R − Landsberg space as *generalized $\mathcal{B}$-recurrent tensor* (briefly $G\mathcal{B}-R$).

Let us consider a G$\mathcal{B}$K − R − Landsberg space.

We have the identity [14]

(4.2) $(K_{hijk} - K_{jkhi})y^j = C_{rhk}H^r_i - C_{rik}H^r_h - C_{rhi}H^r_k$.

Using (1.8) in (4.2), we get

(4.3) $(R_{hijk} - R_{jkhi} - C_{hir}H^r_{jk} + C_{jkr}H^r_{hi})y^j = C_{rhk}H^r_i - C_{rik}H^r_h - C_{rhi}H^r_k$.

Taking the covariant derivative for (4.3) with respect to x^m in the sense of Berwald, using (1.4a), (1.5), (1.11), the symmetric property of the (h)hv-torsion C_{ijk} in all its indices and in view of (1.12), we get

(4.4) $\lambda_m(R_{hijk} - R_{jKhi})y^j = \mathcal{B}_m(C_{rhk}H^r_i - C_{rik}H^r_h)$.

Using (1.8), (1.4a) and (1.11) in (4.4), we get

(4.5) $\lambda_m(K_{hijk} - K_{jkhi})y^j - \lambda_m C_{hir}H^r_k = \mathcal{B}_m(C_{rhk}H^r_i - C_{rik}H^r_h)$.

Using (4.2) in (4.5) and the symmetric property of the (h)hv-torsion C_{ijk} in all its indices, we get

$$\mathcal{B}_m(C_{rhk}H^r_i - C_{rik}H^r_h) = \lambda_m(C_{rhk}H^r_i - C_{rik}H^r_h) - 2\lambda_m C_{hir}H^r_k\ .$$

This shows that

(4.6) $\mathcal{B}_m(C_{rhk}H^r_i - C_{rik}H^r_h) = \lambda_m(C_{rhk}H^r_i - C_{rik}H^r_h)$

If and only if

$$C_{hir}H^r_k = 0.$$

Thus, we conclude

Theorem 4.1. *In $G\mathcal{B}K - R - Landsberg$ space, the tensor $(C_{rhk}H^r_i - C_{rik}H^r_h)$ behaves as recurrent, if and only if $C_{hir}H^r_k = 0$.*

We have the identity [14]

(4.7) $K_{ijhk} + K_{ikjh} + K_{ihkj} = -2(C_{ijr}H^r_{hk} + C_{ikr}H^r_{jh} + C_{ihr}H^r_{kj})$.

Taking the covariant derivative for (4.7) with respect to x^m in the sense of Berwald, we get

(4.8) $\mathcal{B}_m(K_{ijhk} + K_{ikjh} + K_{ihkj}) = -2\mathcal{B}_m(C_{ijr}H^r_{hk} + C_{ikr}H^r_{jh} + C_{ihr}H^r_{kj})$.

Transvecting the condition (2.1) by g_{ir} , using (1.9), (3.3a), (1.2a) and the symmetric property of the metric tensor g_{ij} , we get the condition

(4.9) $\mathcal{B}_m K_{rjkh} = \lambda_m K_{rjkh} + \mu_m(g_{hr}g_{jk} - g_{kr}g_{jh})$.

In view of the condition (4.9) and by using the property of the symmetric property of the metric tensor g_{ij} in (4.8), we get

(4.10) $\lambda_m(K_{ijhk} + K_{ikjh} + K_{ihkj}) = -2\mathcal{B}_m(C_{ijr}H^r_{hk} + C_{ikr}H^r_{jh} + C_{ihr}H^r_{kj})$.

Using (4.7) in (4.10), we get

(4.11) $\mathcal{B}_m(C_{ijr}H^r_{hk} + C_{ikr}H^r_{jh} + C_{ihr}H^r_{kj}) = \lambda_m(C_{ijr}H^r_{hk} + C_{ikr}H^r_{jh} + C_{ihr}H^r_{kj})$.

Transvecting (4.11) by g^{is} , using (1.4c) and (3.3b), we get

(4.12) $\mathcal{B}_m(C^s_{jr}H^r_{hk} + C^s_{kr}H^r_{jh} + C^s_{hr}H^r_{kj}) = \lambda_m(C^s_{jr}H^r_{hk} + C^s_{kr}H^r_{jh} + C^s_{hr}H^r_{kj})$.

Transvecting (4.11) by y^h , using (1.11), (1.4a) and (1.5), we get

(4.13) $\mathcal{B}_m(C_{ijr}H^r_k - C_{ikr}H^r_j) = \lambda_m(C_{ijr}H^r_k - C_{ikr}H^r_j)$.

Thus, we conclude

Theorem 4.2. *In $G\mathcal{B}K - R - Landsberg$ space, the tensors $(C_{ijr}H^r_{hk} + C_{ikr}H^r_{jh} + C_{ihr}H^r_{kj})$, $(C^s_{jr}H^r_{hk} + C^s_{kr}H^r_{jh} + C^s_{hr}H^r_{kj})$ and $(C_{ijr}H^r_k - C_{ikr}H^r_j)$ behave as recurrent.*

Taking the covariant derivative for (4.7) with respect to x^m in the sense of Berwald, we get

(4.14) $\mathcal{B}_m(K_{ijhk} + K_{ikjh} + K_{ihkj}) = -2\{(\mathcal{B}_m C_{ijr})H^r_{hk} + C_{ijr}(\mathcal{B}_m H^r_{hk}) +$
$+(\mathcal{B}_m C_{ikr})H^r_{jh} + C_{ikr}(\mathcal{B}_m H^r_{jh}) + (\mathcal{B}_m C_{ihr})H^r_{kj} + C_{ihr}(\mathcal{B}_m H^r_{kj})$.

Using (4.8), (1.12) and (4.11) in (4.14), we get

(4.15) $(\mathcal{B}_m C_{ijr})H^r_{hk} + (\mathcal{B}_m C_{ikr})H^r_{jh} + (\mathcal{B}_m C_{ihr})H^r_{kj} = 0$.

Transvecting (4.15) by y^j, y^k and y^h ,separately, using (1.4a), (1.5) and (1.12), we get

(4.16) $(\mathcal{B}_m C_{ikr})H^r_h - k/h = 0$,

(4.17) $(\mathcal{B}_m C_{ihr})H^r_j - h/j = 0$

and

(4.18) $(\mathcal{B}_m C_{ijr})H^r_k - j/k = 0$,

respectively.

Thus, we conclude

Theorem 4.3. *In $G\mathcal{B}K - R - Landsberg$ space, we have the identities (4.15), (4.16), (4.17) and (4.18).*

5. Projection On Indicatrix For SomeTensors

Let us consider a $G\mathcal{B}K - R -$ affinely connected space for which the associate curvature tensor R_{ijkh} is recurrent in the sense of Berwald, i. e. satisfied the condition (3.4).

In view of (1.16), the projection of the associate curvature tensor R_{ijkh} on indicatrix is given by

(5.1) $p\,.\,R_{ijkh} = R_{abcd}h_i^a h_j^b h_k^c h_h^d.$

Taking the covariant derivative for (5.1) with respect to x^m in the sense of Berwald ,we get

(5.2) $\mathcal{B}_m(p\,.\,R_{ijkh}) = \mathcal{B}_m(R_{abcd}h_i^a h_j^b h_k^c h_h^d).$

Using the condition (3.4) and the fact h_β^α is covariant constant in (5.2), we get

(5.3) $\mathcal{B}_m(p\,.\,R_{ijkh}) = \lambda_m R_{abcd}h_i^a h_j^b h_k^c h_h^d.$

Using (5.1) in (5.3), we get

$$\mathcal{B}_m(p\,.\,R_{ijkh}) = \lambda_m(p\,.\,R_{ijkh}).$$

Thus, we conclude

Theorem 4.2.2. *In* $G\mathcal{B}K - R -$ *affinely connected space, the projection of the associate curvature tensor* R_{ijkh} *on indicatrix is recurrent in the sense of Berwald if and only if (3.5) holds good.*

Let us consider a $G\mathcal{B}K - R -$Landsberg space for which the tensor $(C_{rhk}H_i^r - C_{rik}H_h^r)$ is recurrent in the sense of Berwald, i. e. satisfied the (4.6).

In view of (1.16), the projection of the tensor $(C_{rhk}H_i^r - C_{rik}H_h^r)$ on indicatrix is given by

(5.4) $p\,.\,(C_{rhk}H_i^r - C_{rik}H_h^r) = (C_{abc}H_d^a - C_{adc}H_b^a)h_r^a h_h^b h_k^c h_i^d h_a^r.$

Taking the covariant derivative for (5.4) with respect to x^m in the sense of Berwald, we get

(5.5) $\mathcal{B}_m[p\,.\,(C_{rhk}H_i^r - C_{rik}H_h^r)] = \mathcal{B}_m[(C_{abc}H_d^a - C_{adc}H_b^a)h_r^a h_h^b h_k^c h_i^d h_a^r].$

Using (4.6) and the fact h_β^α is covariant constant in(5.5), we get

(5.6) $\mathcal{B}_m[p\,.\,(C_{rhk}H_i^r - C_{rik}H_h^r)] = \lambda_m(C_{abc}H_d^a - C_{adc}H_b^a)h_i^a h_j^b h_k^c h_h^d.$

Using (5.4) in (5.6), we get

$$\mathcal{B}_m[p\,.\,(C_{rhk}H_i^r - C_{rik}H_h^r)] = \lambda_m[p\,.\,(C_{rhk}H_i^r - C_{rik}H_h^r)].$$

Thus, we conclude

Theorem 4.2.3. *In* $G\mathcal{B}K - R -$ *Landsberg space, the projection of the tensor* $(C_{rhk}H_i^r - C_{rik}H_h^r)$ *on indicatrix is recurrent in the sense of Berwald if and only if* $C_{hir}H_k^r = 0$.

Let us consider a $G\mathcal{B}K - R -$Landsberg space for which the tensor $(C_{ijr}H_{hk}^r + C_{ikr}H_{jh}^r + C_{ihr}H_{kj}^r)$ is recurrent in the sense of Berwald. i. e. satisfied (4.11).

In view of (1.16), the projection of the tensor $(C_{ijr}H_{hk}^r + C_{ikr}H_{jh}^r + C_{ihr}H_{kj}^r)$ on indicatrix is given by

(5.7) $p\,.\,(C_{ijr}H_{hk}^r + C_{ikr}H_{jh}^r + C_{ihr}H_{kj}^r) = (C_{abc}H_{de}^c + C_{aec}H_{bd}^c$
$+C_{adc}H_{eb}^c)h_i^a h_j^b h_r^c h_c^r h_d^h h_e^k.$

Taking the covariant derivative for (5.7) with respect to x^m in the sense of Berwald, we get

(5.8) $\mathcal{B}_m[p\,.\,(C_{ijr}H_{hk}^r + C_{ikr}H_{jh}^r + C_{ihr}H_{kj}^r)] = \mathcal{B}_m[(C_{abc}H_{de}^c + C_{aec}H_{bd}^c$
$+C_{adc}H_{eb}^c)h_i^a h_j^b h_r^c h_c^r h_d^h h_e^k].$

Using (4.11) and the fact h_β^α is covariant constant in (5.8), we get

(5.9) $\mathcal{B}_m p\,.\,[(C_{ijr}H_{hk}^r + C_{ikr}H_{jh}^r + C_{ihr}H_{kj}^r)] = \lambda_m(C_{abc}H_{de}^c + C_{aec}H_{bd}^c$
$+C_{adc}H_{eb}^c)h_i^a h_j^b h_r^c h_c^r h_d^h h_e^k.$

Using (5.7) in (5.9), we get

$$\mathcal{B}_m[p\,.\,(C_{ijr}H_{hk}^r + C_{ikr}H_{jh}^r + C_{ihr}H_{kj}^r)] = \lambda_m[p\,.\,(C_{ijr}H_{hk}^r + C_{ikr}H_{jh}^r$$

$+C_{ihr}H^r_{kj})]$.

Thus, we conclude

Theorem 4.2.4. *In* $G\mathcal{B}K-R-$ *Landsberg space, the projection of the tensor* $(C_{ijr}H^r_{hk} + C_{ikr}H^r_{jh} + C_{ihr}H^r_{kj})$ *on indicatrix is recurrent in the sense of Berwald.*

Let us consider a $G\mathcal{B}K-R-$ Landsberg space for which the tensor $(C^s_{jr}H^r_{hk} + C^s_{kr}H^r_{jh} + C^s_{hr}H^r_{kj})$ is recurrent in the sense of Berwald, i. e. satisfied (4.12).

In view of (1.16), the projection of the tensor $(C^s_{jr}H^r_{hk} + C^s_{kr}H^r_{jh} + C^s_{hr}H^r_{kj})$ on indicatrix is given by

(5.10) $p\,.(C^s_{jr}H^r_{hk} + C^s_{kr}H^r_{jh} + C^s_{hr}H^r_{kj}) = (C^a_{bc}H^c_{de} + C^a_{ec}H^c_{bd} + C^a_{de}H^c_{eb})h^s_a h^b_j h^c_r h^r_c h^h_d h^k_e$.

Taking the covariant derivative for (5.10) with respect to x^m in the sense of Berwald, we get

(5.11) $\mathcal{B}_m p\,.(C^s_{jr}H^r_{hk} + C^s_{kr}H^r_{jh} + C^s_{hr}H^r_{kj}) = \mathcal{B}_m(C^a_{bc}H^c_{de} + C^a_{ec}H^c_{bd} + C^a_{de}H^c_{eb})h^s_a h^b_j h^c_r h^r_c h^h_d h^k_e$.

Using (4.12) the fact h^α_β is covariant constant in (5.11), we get

(5.12) $\mathcal{B}_m p\,.(C^s_{jr}H^r_{hk} + C^s_{kr}H^r_{jh} + C^s_{hr}H^r_{kj}) = \lambda_m(C^a_{bc}H^c_{de} + C^a_{ec}H^c_{bd} + C^a_{de}H^c_{eb})h^s_a h^b_j h^c_r h^r_c h^h_d h^k_e$.

Using (5.10) in (5.12), we get

$$\mathcal{B}_m[p\,.\left(C^s_{jr}H^r_{hk} + C^s_{kr}H^r_{jh} + C^s_{hr}H^r_{kj}\right)] = \lambda_m[p\,.(C^s_{jr}H^r_{hk} + C^s_{kr}H^r_{jh} + C^s_{hr}H^r_{kj})].$$

Thus, we conclude

Theorem 4.2.5. *In* $G\mathcal{B}K-R-$ *Landsberg space, the projection of the tensor* $\left(C^s_{jr}H^r_{hk} + C^s_{kr}H^r_{jh} + C^s_{hr}H^r_{kj}\right)$ *on indicatrix is recurrent in the sense of Berwald.*

Let us consider a $G\mathcal{B}K-R-$ Landsberg space for which tensor $(C_{ijr}H^r_k - C_{ikr}H^r_j)$ is recurrent in the sense of Berwald, i. e. satisfied (4.13).

In view of (1.16), the projection of the tensor $(C_{ijr}H^r_k - C_{ikr}H^r_j)$ on indicatrix is given by

(5.13) $p\,.(C_{ijr}H^r_k - C_{ikr}H^r_j) = (C_{abc}H^c_d - C_{adc}H^c_b)h^a_i h^b_j h^c_r h^r_c h^d_k$.

Taking the covariant derivative for (5.13) with respect to x^m in the sense of Berwald, we get

(5.14) $\mathcal{B}_m p\,.\left(C_{ijr}H^r_k - C_{ikr}H^r_j\right) = \mathcal{B}_m(C_{abc}H^c_d - C_{adc}H^c_b)h^a_i h^b_j h^c_r h^r_c h^d_k$.

Using (4.13) the fact h^α_β is covariant constant and in (5.14), we get

(5.15) $\mathcal{B}_m p\,.(C_{ijr}H^r_k - C_{ikr}H^r_j) = \lambda_m(C_{abc}H^c_d - C_{adc}H^c_b)h^a_i h^b_j h^c_r h^r_c h^d_k$

Using (4.13) in (5.15), we get

$$\mathcal{B}_m\left[p\,.\left(C_{ijr}H^r_k - C_{ikr}H^r_j\right)\right] = \lambda_m.\left[p\,.\left(C_{ijr}H^r_k - C_{ikr}H^r_j\right)\right].$$

Thus, we conclude

Theorem 4.2.6. *In* $G\mathcal{B}K-R-$ *Landsberg space, the projection of the tensor* $\left(C_{ijr}H^r_k - C_{ikr}H^r_j\right)$ *on indicatrix is recurrent in the sense of Berwald.*

REFERENCES

Berwald, L.: On Finsler and Cartan geometries III, Two dimensioned Finsler spaces will rectilinear external, Ann. of Math., (2) 42, (1941), 84-122.

Berwald, L.: Über Finslersche und Cartansghe geometrie IV. Projktiv- krümmung allgemeiner affiner räume und Finslersche räume skalarer, krümmung, Ann. Math.,(2) 48, (1947), 755-781.

Cartan, É.: *Les espaces de Finsler*, Actualités, Paris, 79, (1934) ; 2nd edit. (1971).

Izumi, H.: On $^*P$ -Finsler space I, Memo. Defence Acad. Japan, 16, (1976), 133-138.

Izumi, H.: On $^*P$ -Finsler space II, Memo. Defence Acad. Japan, 17, (1977), 1-9.

Izumi, H.: On $^*P$ -Finsler space of scalar curvature, Tensor N.S., 38, (1982), 220-222.

Landsberg, G.: Über die krumming in der variation rechnung, Math. Ann., 65, (1908), 313 349.

Matsumoto, M.: On C-reducible Finsler space, M. S., 24, (1972), 92-137.

Matsumoto, M.: Remarks on Berwald and Landsberg spaces, Contemporary Math. Vol. 195, (1996), 79-81.

Muhib, **A. A. A.**: On independent components of tensor, I-relative tensor and R^h-generalized trirecurrent Finsler spaces, M. Sc. Thesis, University of Aden, (Aden) (Yemen), (2009).

Pande, H.D and **Singh, B.**: On existence of affinely connected Finsler space with recurrent tensor field, Reprinted from Indian Journal of Pure Applied Mathematics, Vol. 8, No. 3, (March 1977), 295-301.

Qasem, **F. Y. A.** and **Baleedi**, **S. M. S.**: On a generalized $\mathcal{B}K$ – recurrent Finsler spaces, International Journal of Sciences : Basic and Applied Research, Volume 28, No. 3, (2016),195 -203.

Qasem, **F. Y. A.** and **Baleedi**, **S. M. S.**:Necessary and sufficint condition for generalized recurrent tensor, Imperial Journal of Interdisciplinary Research,Vol.2,Issu 11,(2016)**,**769-**77**6.

Rund, **H.**: *The differential geometry of Finsler spaces*, Springer-Verlag, Berlin Göttingen Heidelberg, (1959); 2nd Edit. (in Russian), Nauka, (Moscow), (1981).

Verma, **R.**: *Some transformation in Finsler spaces*, D.Phil. Thesis, University of Allahabad, (Allahabad) (India), (1991).

Yasuda, H.: On Landsberg space, Tensor N. S., Vol. 34, (1980), 76-84.